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SMALL CANCELLATION THEORY WITH A WEAKENED SMALL CANCELLATION HYPOTHESIS. I. THE BASIC THEORY

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ABSTRACT
We introduce a new small cancellation condition, the condition W(4), which weakens the condition C(4) & T(4), and develop the corresponding small cancellation theory. In this paper we develop the basic theory and prove a version of Greendlinger's Lemma. In the forthcoming papers with the same title we prove the word problem and the conjugacy problem for groups having a presentation which satisfies the condition W(4).

Introduction

The objective of this and the forthcoming articles with the same title is to introduce a weakened small cancellation condition which makes the theory substantially more flexible than classical Small Cancellation Theory.

The motivation for weakening small cancellation hypotheses can be explained by the following example: Let $\mathcal{G}$ be a finitely generated group defined by the single relator $[A,B,B]=1$, where $A,B\in\mathcal{G}$, $[A,B]=A^{-1}B^{-1}AB$ and $[A,B,B]=[[A,B],B]$. We would like to study this group through its presentation and in particular to solve the conjugacy problem for it. Now, if A and B are generators of $\mathcal{G}$ $(A\neq B)$ then it follows easily that this presentation of $\mathcal{G}$ satisfies the condition C(4) & T(4), hence by the classical theory it has solvable conjugacy problem. Though $\mathcal{G}$ resembles a small cancellation group even for the case when A and B are not generators, it satisfies neither C(4) & T(4) nor C(6). For it may

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very well happen that in a diagram M, which represents a word equal to $1_{\mathscr{G}}$, there are two regions, say D_1 and D_2, such that D_1 has many neighbours but its boundary also has vertices with valency 3, while D_2 has exactly 4 neighbours and every vertex on the boundary of D_2 has valency at least 4. Thus our problem is to formulate a small cancellation condition which incorporates the condition C(4) & T(4) and some of the conditions C(p) for some p.

This we resolve by introducing a common generalization of the conditions C(4) & T(4) and the condition C(8) with 3 interim conditions in such a way that if a region has "few" neighbours then this fact is compensated by the requirement that some of the vertices on its boundary have valency at least 4. (For the precise definition see 1.1.) We call this condition "Weak C(4) & T(4)" or briefly condition W(4).

In this paper we develop the basic theory for diagrams which satisfy the condition W(4) and in [1] and [2] we solve the word and conjugacy problems respectively.

This W(4) theory is particularly suitable for studying some finitely generated groups defined by relators which are commutators of sufficiently high weight. For example, finitely generated groups defined by a single relator which is a commutator of weight $\geqq 4$ have solvable conjugacy problem, as they have a presentation which satisfies the condition W(4). (See [3], [4] and [5].) For the same reason several other finitely generated groups have solvable word and conjugacy problems. (See [6] and [7].) In some cases it is relatively easy to detect that a group has a W(4) presentation. (See [6].)

The main result of this paper is the following version of Greendlinger's Lemma for W(4) maps.

THEOREM A. *Let M be a connected, simply connected map which satisfies the Condition* W(4) *and contains more than one region. For every boundary region D of M denote by $i(D)$ the number of inner edges of D. Then M has a set $\mathscr{D}$ of boundary regions with $i(D)\leqq 3$ such that the following conditions hold*:

(I) *$\partial D \cap \partial M$ is connected for every $D \in \mathscr{D}$*;

(II) *if $i(D)=3$ then ∂D has an inner vertex with valency* 3;

(III) $\Sigma_{D\in\mathscr{D}}[4-i(D)]\geqq 6$.

Let us outline very briefly the proof of the theorem.

We associate to every inner region D of a map M a number $S(D)$ which may be regarded as a measure of the "curvature" around D. We show that if $S(D)\leqq 0$ for every inner region D then the theorem holds by arguments from the classical theory. We call such maps G(3). At this point we introduce separating paths which are the main tool of the theory, and using them to define some reduction process from general W(4) maps to G(3) maps.

The paper is organized as follows. Let $\mathcal{G} = \langle x \mid \mathcal{R} \rangle$ be a presentation of $\mathcal{G}$ with $\mathcal{R}$ symmetrized.

In section 1 we define the condition W(4) and indicate how this condition can be checked in $\mathcal{R}$. In section 2 we prove the basic properties of W(4) maps, namely that in a connected and simply connected map which satisfies the condition W(4) the closure of every region is simply connected and the common boundary of two different regions is connected. In section 3 we deal with maps which satisfy the condition G(3).

In section 4 separating paths are introduced. We prove with the aid of the results in section 3 that every simply connected W(4) map with connected interior which doesn't satisfy G(3) does have separating paths. This is the central result of the whole paper.

In section 5 we prove our main Theorem.

§1. The condition W(4)

In this paper we follow the notation of [8]. Also we refer to [8] for unexplained terms. In what follows we shall mean by a connected and simply connected map a simply connected map with a connected interior.

1.1. Definition. Let M be a map. We say that M satisfies the condition W(4) if for every region D in M for which $\partial D \cap \partial M$ doesn't contain an edge one of the following holds (see Fig. 1):

(i) $d(D) = 4$ and $d(v) \geqq 4$ for every vertex v on ∂D.
(ii) $d(D) = 5$ and $d(v) = 4$ for at least 4 vertices v of ∂D.
(iii) $d(D) = 6$ and $d(v) = 4$ for at least 3 vertices v of ∂D.
(iv) $d(D) = 7$ and $d(v) = 4$ for at least 2 vertices v of ∂D.
(v) $d(D) \geqq 8$.

1.2. The condition W(4) is defined in terms of the number of neighbours $d(D)$ and the valency of the vertices of ∂D. Let us denote by $a(D)$ the number of edges of D which has an endpoint with valency 3. Then the following proposition shows that the condition W(4) implies a kind of "average" C(4) & T(4) condition.

Proposition. *Let M be a* W(4) *map and let D be a region of M such that $\partial D \cap \partial M$ doesn't contain an edge. If $\bar{D}$ is simply connected then* $d(D) - \frac{1}{3}a(D) \geqq 4$.

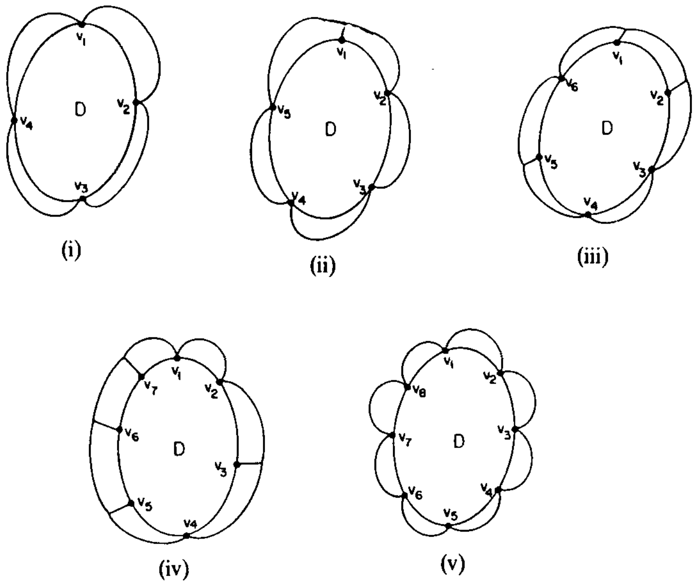


Fig. 1.

PROOF. Immediate by Definition 1.1.

1.2. REMARK. In practice one is interested in algebraic conditions on the relators which ensure the condition W(4) on the corresponding diagrams. Since the condition W(4) is defined in terms of the number of edges of an inner region and in terms of the valency of its boundary vertices, we can easily formulate such a condition, if we can interpret these geometrical terms algebraically. But such an interpretation is clear. (See [8], p. 242.)

§2. Basic properties of maps with condition W(4)

2.1. Simply connected maps which satisfy one of the classical small cancellation conditions have the nice properties that the boundary of every region of it is a simple curve and if two regions have a nontrivial common boundary, then it is connected. (See [9].) The main result of this section shows that maps with condition W(4) also share these properties.

2.1. THEOREM. *Let M be a connected, simply connected map which contains more than one region. If M satisfies the condition* W(4) *then*

(a) *∂D is a simple curve for every region D of M.*

(b) *If D_1 and D_2 are regions of M such that $\partial D_1 \cap \partial D_2 \neq \emptyset$ then $\partial D_1 \cap \partial D_2$ is a connected simple curve.*

(c) *Every connected and simply connected submap* M_0 *of* M *with a simple boundary and without boundary edges on* ∂M *has at least* 4 *neighbours. Here a region* D *is a neighbour of* M_0 *if* $\partial D \cap \partial M_0$ *contains an edge.*

The proof of the theorem follows the ideas of [9].

In order to make things more transparent we introduce below a notion of convexity. This notion is due to E. Rips and it did not appear in the original version. I am indebted to Professor Rips for his suggestion.

DEFINITION. Let M be a map. M satisfies the condition $CN(k)$ if for every k regions $D_1, \ldots, D_k$ in M every component of $\bar{D}_1 \cup \cdots \cup \bar{D}_k$ is simply connected.

Clearly $CN(k) \Rightarrow CN(l)$ for $l \leqq k$. Our theorem can be rephrased as follows:

THEOREM. *Let* M *be a connected and simply connected* W(4) *map. Then* M *satisfies* $CN(3)$.

Recall that for a region D we denote by $d(D)$ the number of edges of D. For every vertex v in a connected and simply connected submap M_0 of M we denote by $d_0(v)$ the valency of v in M_0.

2.2. We prove the theorem through the following lemma.

2.2. LEMMA. *Let* M *be a connected map and let* M_0 *be a connected simply connected submap of* M *which has no boundary edges on* ∂M. *Assume that*

(i) M *satisfies the condition* W(4),

(ii) *every region of* M_0 *has a simple boundary.*

Then

(a) $\Sigma^{\cdot}_{M_0}[3 - d_0(v)] \geqq 4$, *where the summation is over the boundary vertices of* M_0.

(b) ∂M_0 *contains at least* 4 *vertices with valency* 2 *in* M_0.

PROOF. (a) By the basic formula (3.2) in [8, p. 243] with $p = q = 4$

$$\sum_{M_0}^{\cdot} [3 - d_0(v)] + \sum_{M_0}^{\circ} [4 - d_0(v)] + \sum_{M_0} [4 - d(D)] \geqq 4.$$

Hence it is enough to show that $\Sigma^{\circ}_{M_0}[4 - d_0(v)] + \Sigma_{M_0}[4 - d(D)] \leqq 0$. As $4 - d_0(v) \leqq 0$ for every v with $d_0(v) \geqq 4$, we shall prove $\Sigma_{v \in V}[4 - d_0(v)] + \Sigma[4 - d(D)] \leqq 0$. Here V is the set of all the inner vertices of M_0 with valency 3. Denote

$$\delta = \sum_{v \in V} [4 - d_0(v)] + \sum_{M_0} [4 - d(D)]$$

and denote $\mathcal{D} = \bigcup_{v \in V} N(v)$ where $N(v)$ is the set of all the regions of M which contain v on their boundary. Since for every region D in M_0, ∂D contains no edges in ∂M, so $d(D) \geqq 4$ by Definition 1.1 and we have

$$(1) \qquad \delta \leqq \sum_{v \in V} [4 - d_0(v)] + \sum_{D \in \mathcal{D}} [4 - d(D)].$$

Now, for every $v \in V$, $4 - d_0(v) = 1$ holds and for every $v \in V$ there are exactly 3 regions in $\mathcal{D}$ which contain v, as every region of M_0 has a simple boundary by assumption. Consequently $\Sigma_{v \in V} [4 - d_0(v)] = |V|$ and if $a(D)$ is the number of edges of ∂D which have an endpoint in V then $|V| \leqq \frac{1}{3} \Sigma_{D \in \mathcal{D}} a(D)$. Thus

$$(2) \qquad \sum_{v \in V} [4 - d_0(v)] \leqq \frac{1}{3} \sum_{D \in \mathcal{D}} a(D).$$

Substituting (1) in (2) gives

$$(3) \qquad \delta \leqq \sum_{D \in \mathcal{D}} [\tfrac{1}{3} a(D) + 4 - d(D)].$$

But $\frac{1}{3} a(D) + 4 - d(D) = -(d(D) - \frac{1}{3} a(D) - 4) \geqq 0$ by Proposition 1.2. Consequently, (3) implies $\delta \leqq 0$, as required.

(b) If v is a boundary vertex of M_0 then $3 - d_0(v) > 0$ only if $d_0(v) = 2$. Thus the result follows directly from part (a).

Remark. If v is a boundary vertex of M_0 with valency 2 then M_0 has a unique boundary region which contains v on its boundary and because of $CN(1)$ there are at least two regions in M which are not in M_0 and which contain v on their boundary. This leads to the following.

Proposition. *Let M be a connected and simply connected map. Consider the set $\mathcal{S}$ of all the connected and simply connected submaps M_0 with the property that $\partial M \cap \partial M_0$ doesn't contain an edge. If every M_0 in $\mathcal{S}$ contains at least $k+1$ ($k \geqq 1$) vertices with valency 2 on its boundary then M satisfies $CN(k)$.*

Proof. We prove the proposition by induction on k.

$k = 1$: Let D be a region in M. If $\bar{D}$ is not simply connected let $H_1, \ldots, H_r$ be the bounded components of $M \backslash D$. Every H_i is simply connected and there exists an H_i which is connected to exactly one H_j by a boundary edge of ∂D. (See Fig. 2.)

Assume H is such a component. Then H has exactly one boundary vertex w with $d_0(w) = 2$, namely the endpoint of the edge connecting H with another component. But this contradicts the assumption of the proposition.

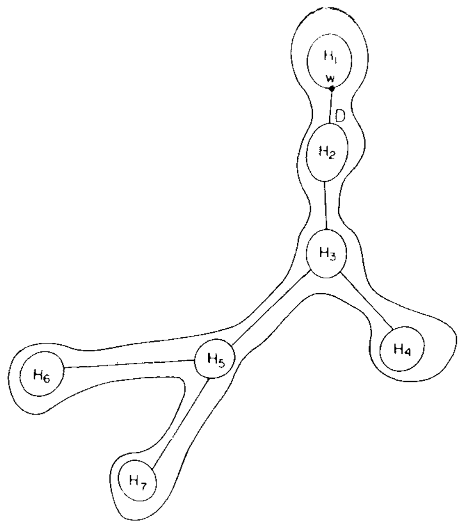
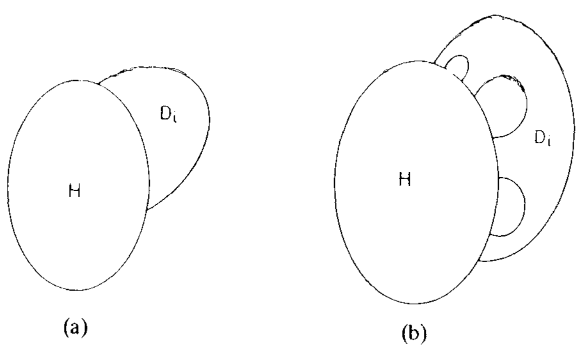


Fig. 2.

$k > 1$: Let $D_1, \ldots, D_k$ be regions in M such that $B = \bar{D}_1 \cup \cdots \cup \bar{D}_k$ is connected. We have to show that B is simply connected. Suppose this is not the case and let H be one of the holes in B. Let $\partial D_i \cap \partial H \neq \emptyset$. Then either $\partial D_i \cap \partial H$ is connected (see Fig. 3a) or $\bar{H} \cup \bar{D}_i$ is not simply connected (see Fig. 3b).



(a) (b)

Fig. 3.

We claim that for every D_i such that $\partial D_i \cap \partial H \neq \emptyset$, $\partial D_i \cap \partial H$ is connected. For if not then $\bar{D}_i \cup \bar{H}$ is not simply connected and each one of the holes of $\bar{D}_i \cup \bar{H}_i$ contains at least one D_j. Let B_0 be obtained from B by deleting all $\bar{D}_j$ such that D_j is contained in some hole of $\bar{D}_i \cup \bar{H}$. Then B_0 is simply connected by the induction hypotheses. On the other hand B_0 necessarily has a hole H_0

such that $H_0 \supseteq H$, a contradiction. Thus $\partial D \cap \partial H$ is connected. Therefore, we can write

$$\partial H = v_0\mu_1 v_1 \mu_1 \cdots v_{l-1}\mu_l v_0, \qquad \text{where } \mu_j = \partial H \cap D_{i_j}$$

(see Fig. 4).

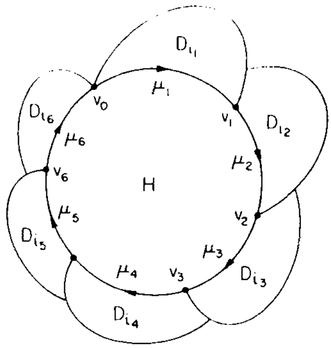


Fig. 4.

Thus $l \leqq k$. Since every $\bar{D}_i$ is simply connected, we see that $v_0, v_1, \ldots, v_{l-1}$ are the only vertices on ∂H for which $d_0(v) = 2$ may hold, a contradiction. Thus B is simply connected, as required.

2.3. Proof of the Theorem. By Lemma 2.2(b) every $M_0 \in \mathcal{S}$ contains at least 4 vertices with valency 2. Consequently, by Proposition 2.2, M satisfies $CN(3)$, as required.

§3. Maps satisfying the condition G(3)

3.1. The Curvature Formula for connected and simply connected C(4) & T(4) maps was derived through the basic formula [8]. The final argument was that since $d(D) \geqq 4$ and $d(v) \geqq 4$ for every inner region D and inner vertex v, (*) below holds.

$$(*) \qquad \delta = \sum{}^{\circ}(4 - d(D)) + \sum{}^{\circ}(4 - d(v)) \leqq 0.$$

However, it is clear that even if the condition C(4) & T(4) doesn't hold but we can guarantee in some way that the positive contribution to δ due to vertices of valency 3 is fully compensated by a negative contribution due to regions which have more than 4 neighbours, we still can satisfy (*). The problem in using this idea is that while (*) imposes a *global* condition on M the conditions on $\mathcal{R}$ imply *local* conditions on the regions. We introduce below an almost local condition which almost implies (*). (See Lemma 3.3.)

3.2. DEFINITIONS. (a) Let M be a map and let D be an inner region of M. For every vertex v of ∂D let $k(v)$ be the number of inner regions which contain v on their boundary (inner star of v). Define

$$S(D)=4-d(D)+\sum_{v\in\partial D}\frac{4-d(v)}{k(v)}.$$

Since D is an inner region clearly $k(v)\geqq 1$ for every $v\in\partial D$, hence $S(D)$ is well defined. $S(D)$ measures the "local curvature" around D.

(b) Let M be a map. We say that *M satisfies the condition* G(3) if the following two conditions hold:

(i) Every inner vertex v with valency 3 is contained on the boundary of at least one inner region;

(ii) $S(D)\leqq 0$ for every inner region D of M.

3.3. LEMMA. *Let M be a map which satisfies the condition* G(3). *Then*

$$\sum{}^{\circ}[4-d(D)]+\sum{}^{\circ}[4-d(v)]\leqq 0.$$

PROOF. Denote by $\mathcal{D}_0$ the set of all the inner regions which contain a vertex with valency 3 on their boundary. For $D\in\mathcal{D}_0$ denote by $V(D)$ the set of vertices with valency 3 on ∂D. Since every inner vertex v with valency 3 is on the boundary of a $D\in\mathcal{D}_0$ by assumption (i) and moreover v is on the boundary of $k(v)$ regions of $\mathcal{D}_0$ we have

$$\sum{}^{\cdot}[4-d(v)]\leqq\sum_{D\in\mathcal{D}_0}\left(\sum_{v\in V(D)}\frac{4-d(v)}{k(v)}\right).\tag{1}$$

On the other hand as $d(D)\geqq 4$, the following clearly holds:

$$\sum{}^{\circ}[4-d(v)]\leqq\sum_{D\in\mathcal{D}_0}(4-d(D)).\tag{2}$$

Combining (1) with (2) we get

$$\sum{}^{\circ}[4-d(D)]+\sum{}^{\circ}[4-d(V)]\leqq\sum_{D\in\mathcal{D}_0}\left[4-d(D)+\sum_{v\in V(D)}\frac{4-d(v)}{k(v)}\right]=\sum_{D\in\mathcal{D}_0}S(D).$$

But by assumption (ii), $S(D)\leqq 0$ for every inner region D. Consequently

$$\sum{}^{\circ}[4-d(D)]+\sum{}^{\circ}[4-d(V)]\leqq 0,$$

as required.

As a corollary we have the following Curvature Theorem for G(3) maps.

PROPOSITION. *Let M be a connected and simply connected map which satisfies the condition* G(3). *Then* $\Sigma^{\circ}(3-i(D))\geqq 4$.

PROOF. The proof is the same as for the case C(4) & T(4). We reproduce it here for sake of completeness. By formula 3.2 in [8, p. 243] we have with $p=q=4$

$$\sum^{\bullet}[3-d(v)]+\sum^{\circ}[4-d(v)]+\sum[4-d(D)]\geqq 4.$$

Passing to the dual map this gives

$$\sum^{\bullet}[3-i(D)]+\sum^{\circ}[4-d(D)]+\sum^{\circ}[4-d(v)]\geqq 4,$$

i.e.

$$\sum^{\bullet}[3-i(D)]+\delta\geqq 4.$$

But $\delta\leqq 0$. Hence $\Sigma^{\bullet}[3-i(D)]\geqq 4$, as required.

The condition W(4) doesn't imply the condition G(3), i.e., in a map satisfying W(4) there can occur inner regions D with $S(D)>0$. Nevertheless, as we shall see, for most inner regions of a map satisfying the condition W(4) there holds $S(D)\leqq 0$. Namely

3.4. LEMMA. *Let M be a connected, simply connected* W(4) *map and let D be an inner region in M. If every vertex with valency* 3 *on* ∂D *lies on the boundary of at most one boundary region then* $S(D)\leqq 0$. *In particular, if D has no common boundary with any boundary region then* $S(D)\leqq 0$.

PROOF. By $CN(1)$ the number of vertices on ∂D is $d(D)$. Consequently $S(D)\leqq 4-d(D)+\frac{1}{2}d(D)$. If $d(D)\geqq 8$ then $4-d(D)+\frac{1}{2}d(D)=4-\frac{1}{2}d(D)\leqq 0$, hence $S(D)\leqq 0$. If $d(D)=7$ then we may have at most 5 vertices with valency 3 each contributing $\frac{1}{2}$ at most to $S(D)$. Thus $S(D)\leqq 4-7+\frac{1}{2}\cdot 5\leqq 0$.

If $d(D)=6$ then ∂D has at most 3 vertices with valency 3 hence $S(D)\leqq 4-6+\frac{1}{2}\cdot 3=-\frac{1}{2}<0$.

If $d(D)=5$ then $S(D)\leqq 4-5+\frac{1}{2}\cdot 1=-\frac{1}{2}<0$. Finally for $d(D)=4$, $S(D)=4-4+0=0$.

3.5. In the situation described by the above lemma, $k(v)\geqq 2$ holds for every inner vertex with valency 3, hence $S(D)\leqq 4-\frac{1}{2}d(D)$. The following proposition, which is a key result for our theory, shows that an inner region D of M with $S(D)>0$ occurs in a rather specific situation only.

PROPOSITION. *Let M be a connected and simply connected map satisfying the condition* W(4). *Let D be an inner region of M such that* $S(D) > 0$. *Then there exists an inner vertex v of valency* 3 *and* 3 *distinct boundary regions* D_1, D_2 *and* D_3 *such that* $v \in \partial D \cap \partial D_1 \cap \partial D_2$ *and* $\partial D \cap \partial D_i$ *contains an edge for* $i = 1,2,3$. *Moreover, if* $\mu_3 = \partial D \cap \partial D_3$ *then* $o(\mu_3)$ *has valency* 3 *in M* (see Fig. 5).

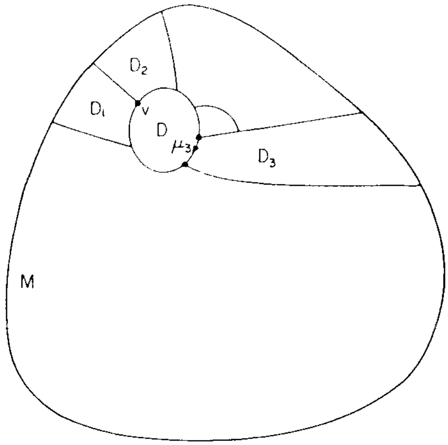


Fig. 5.

PROOF. If $d(D) = 4$ (i.e. every vertex on ∂D has valency at least 4) then certainly the condition W(4) implies $S(D) \leqq 0$. Thus we have only to check the cases $d(D) = 5$, 6, 7 and $\geqq 8$. Since M satisfies $CN(2)$ by Theorem 2.2(b), ∂D has a decomposition

$$(*) \qquad \partial D = v_1 e_1 v_2 e_2 \cdots v_d e_d v_1, \qquad \text{where } d = d(D),$$

such that if e_i is on the common boundary of D and a region E_i (see Fig. 6), then $i \neq j$ implies $E_i \neq E_j$.

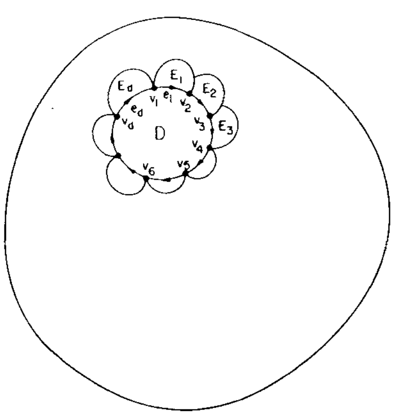


Fig. 6.

By the assumption $S(D) > 0$. By Lemma 3.4 there is a vertex, say v_1, on ∂D with valency 3, such that v_1 is common to two boundary regions E_d and E_1. Consequently $k(v_1) = 1$. Assume now the proposition is false. Then E_d and E_1 are the only boundary regions of M which are neighbours of D (see Fig. 7). Hence $k(v_2) \geqq 2$, $k(v_d) \geqq 2$ and $k(v_t) \geqq 3$ for every t, $3 \leqq t \leqq d-1$. Let us show that this implies $S(D) \leqq 0$ in all four cases.

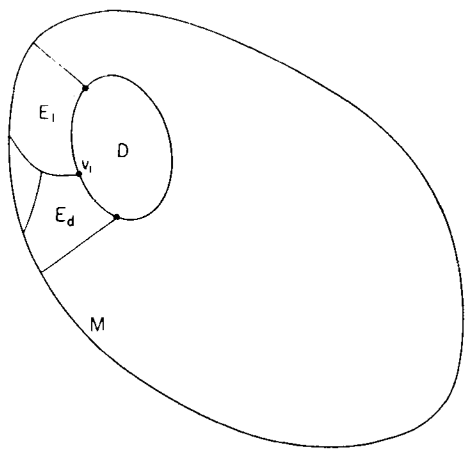


Fig. 7.

d(D) = 5. Then $d(v_i) \geqq 4$ for $t \neq 1$ hence

$$S(D) = (4-5) + \frac{4-3}{k(v_1)} + \sum_{i=2}^{5} \frac{4-d(v_i)}{k(v_i)} \leqq -1 + \frac{1}{1} + 0 = 0.$$

d(D) = 6.

$$S(D) = (4-6) + \frac{4-3}{k(v_1)} + \sum_{i=2}^{6} \frac{4-d(v_i)}{k(v_i)} = -1 + \sum_{i=2}^{6} \frac{4-d(v_i)}{k(v_i)}.$$

We show that

$$\sum_{i=2}^{6} \frac{4-d(v_i)}{k(v_i)} \leqq 1.$$

By the W(4) condition at most two vertices, say v_t and v_s, may have valency 3. Hence

$$\sum_{i=2}^{6} \frac{4-d(v_i)}{k(v_i)} \leqq \frac{4-d(v_t)}{k(v_t)} + \frac{4-d(v_s)}{k(v_s)} = \frac{1}{k(v_t)} \leqq 1,$$

as $k(v_t), k(v_s) \geqq 2$, by assumption.

$d(D) = 7$.

$$S(D) = (4-7) + \frac{4-3}{k(v_1)} + \sum_{i=2}^{7} \frac{4-d(v_i)}{k(v_i)} = -2 + \sum_{i=2}^{7} \frac{4-d(v_i)}{k(v_i)}.$$

By the W(4) condition at most 5 of the vertices $v_2, \ldots, v_7$ may have valency 3 and by the assumption at most 2, say v_s and v_t, $2 \leqq s, t \leqq 7$, may have $k(v_t), k(v_s) \geqq 2$ while the remaining vertices v_l will have $k(v_l) \geqq 3$. Consequently,

$$\sum_{i=2}^{7} \frac{4-d(v_i)}{k(v_i)} \leqq \frac{1}{k(v_s)} + \frac{1}{k(v_t)} + \frac{5-2}{3} \leqq \frac{1}{2} + \frac{1}{2} + \frac{3}{3} = 2$$

and $S(D) \leqq 0$.

$d(D) \geqq 8$.

$$S(D) \leqq 4 - d + \tfrac{1}{3}(d-2) + \tfrac{1}{2} + \tfrac{1}{2} < 0.$$

The rest of the proposition is clear from the construction.

§4. Separating paths

In the previous section we have seen that if a map M satisfies the condition G(3) then M has a curvature theorem (Proposition 3.3). On the other hand we have also seen that a map which satisfies the condition W(4) almost satisfies G(3), i.e. $S(D) \leqq 0$ for almost all inner regions D. In this section we introduce the main tool of our theory, namely, the separating paths by which we shall treat the regions with $S(D) > 0$.

4.1. Definition. Let M be a connected and simply connected map and let μ be a simple path in M such that $\mu \cap \partial M = \{o(\mu), t(\mu)\}$. Then M is divided by μ into two submaps, say M_1 and M_2.

(1) μ is called a *separating path of the first kind* with respect to the ordered pair (M_1, M_2) if there exists a region D such that $\mu \subseteq \partial D$. (See Fig. 8.)

(2) μ is called a *separating path of the second kind* with respect to the ordered pair (M_1, M_2) if there exist distinct regions D_1 and D_2 in M_1 such that $\mu = v_0\mu_1v_1\mu_2v_2$, where μ_i is on the boundary of D_i; $i = 1,2$ and v_1 has valency 3. (See Fig. 9.)

(3) μ is called a *separating path of the third kind* with respect to the ordered pair (M_1, M_2) if there exist distinct regions D_1, D_2 and D_3 in M_1 and E in M_2 such that D_2 is an inner region in M and $\mu = v_0\mu_1v_1\mu_2v_2\mu_3v_3\mu_4v_4$, where $\mu_1 \subseteq \partial D_1$, $\mu_2, \mu_3 \subseteq \partial D_2$, $\mu_4 \subseteq \partial D_3$ and either

(a) $\mu_1, \mu_2 \subseteq \partial E$ and $d_M(v_3) = 3$

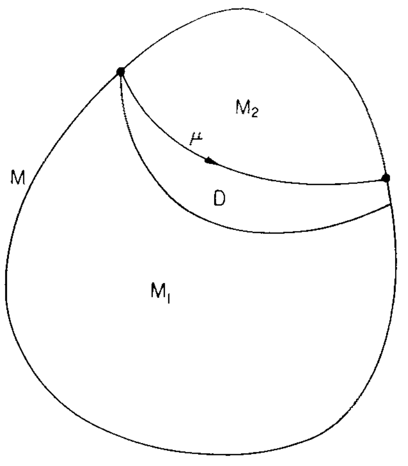


Fig. 8.

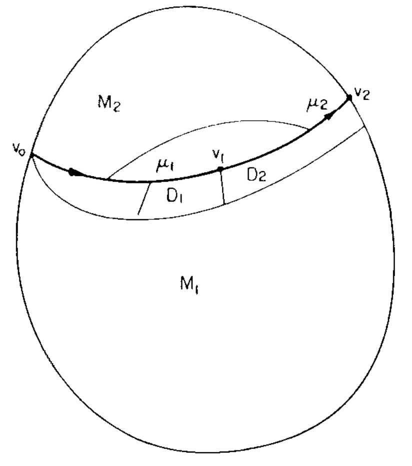


Fig. 9.

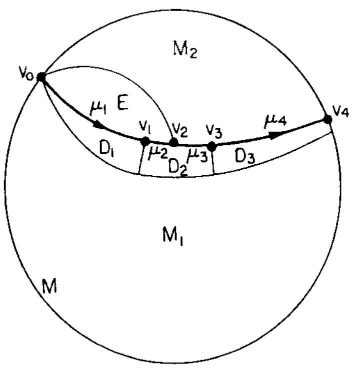
or

(b) $\mu_3, \mu_4 \subseteq \partial E$ and $d_M(v_1) = 3$.

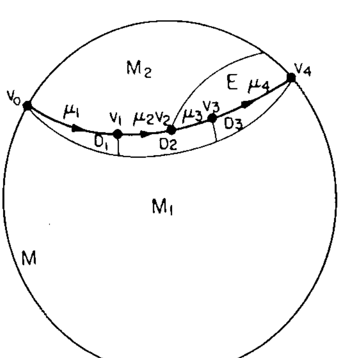
We call M_1 the *core submap* and M_2 the *shell submap*; with respect to μ we shall denote M_1 by Core(μ) and M_2 by Sh(μ). We call the regions D_1, D_2 and D_3 *the associated regions to* μ. (See Fig. 10.)

The main result of this section is the following:

4.2. THEOREM. *Let M be a connected and simply connected map which satisfies the condition* W(4). *If M doesn't satisfy the condition* G(3) *then M has a separating path.*



(a)



(b)

Fig. 10.

PROOF. By Definition 3.1, if M doesn't satisfy G(3) then one of the following cases occur.

Case 1. M has an inner vertex v with valency 3 which is not on the boundary of any inner region.

Case 2. M has an inner region D with $S(D) > 0$.

We shall construct a separating path of the second kind in Case 1 and a separating path of the third kind or of the second kind in Case 2.

Case 1. (See Figs. 11, 12.)

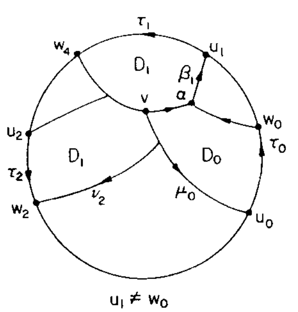
Let D_0, D_1 and D_2 be the boundary regions of M which contain v on their boundary. By the $CN(2)$ property (Theorem 2.1) D_0, D_1 and D_2 are all distinct. Decompose ∂D_i, $i = 0,1,2$, as follows:

$$\partial D_i = v\mu_i u_i \tau_i w_i \nu_i v, \quad \bar{\mu}_i \cap \partial M = u_i \quad \text{and} \quad \bar{\nu}_i \cap \partial M = w_i, \quad i = 0,1,2.$$

1° If $w_i \neq v_{i+1}$ for some i then decompose μ_{i+1} and ν_i by $\mu_{i+1} = \alpha t_i \beta$ and $\nu_i = j t_i \alpha^{-1}$. (Here $\mu_3 = \mu_0$.) Due to the property $CN(2)$ such a decomposition exists and by the assumption $\gamma \neq \varnothing$ and $\beta \neq \varnothing$. We claim that $\mu = w_{i+1}\beta^{-1} t\gamma v_i$ is a separating path of the second kind: It is a simple curve by the properties $CN(2)$ and $CN(1)$; it intersects ∂M exactly in w_{i+1} and v_i, by the definitions of μ_{i+1} and ν_i; β is on the boundary of D_{i+1} and γ is on the boundary of D_i. (Here $D_3 = D_0$.) Finally $d_{\mathrm{Core}(\mu)}(t) = 3$. (See Fig. 11.)

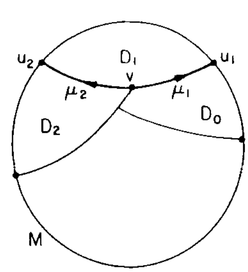
2° If $w_i = v_{i+1}$ for all $i = 0,1,2$ then the above arguments show that $\mu \equiv u_{i+1}\mu_{i+1}^{-1} v \mu_i u_i$ is a separating path of the second kind. (See Fig. 12.)

Case 2. It follows from the definition of $S(D)$ and from Lemma 3.5 that ∂D contains a vertex v with valency 3 and distinct boundary regions D_1, D_2 and D_3 of M such that $\partial D_i \cap \partial D \neq \varnothing$ for $i = 1,2,3$ and either $v \in \partial D_1 \cap \partial D_2$ in which case $0(\partial D_3 \cap \partial D)$ has valency 3 or $v \in \partial D_2 \cap \partial D_3$ and $t(\partial D_1 \cap \partial D)$ has valency



$u_1 \neq w_0$

Fig. 11.



$\mu = u_2 \mu_2^{-1} v \mu_1 u_1$

Fig. 12.

3. Without loss of generality, we shall choose the first possibility. Let $\mu_i = \partial D_i \cap \partial D$ for $i = 1,2,3$. By the properties $CN(1)$ and $CN(2)$ we can decompose ∂D_1 and ∂D_2 as follows (see Figs. 13, 14):

$$\partial D_1 = v\alpha_1 u\beta_1 t_1 \mu_1 v, \qquad \bar{\alpha}_1 \cap \partial M = u,$$

$$\partial D_2 = v\mu_2 t_2 \beta_2 w\alpha_2 v, \qquad \bar{\alpha}_2 \cap \partial M = w.$$

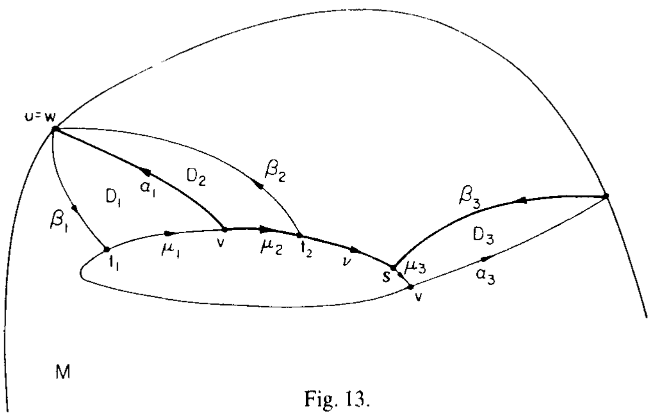


Fig. 13.

Distinguish two cases according as $u = w$ or $u \neq w$.

1° $u = w$ (see Fig. 13). Then $\alpha_1 = \alpha_2^{-1}$. Let $o(\mu_3) = s$ and let ν be the shortest boundary path of D with $0(\nu) = v$ and $t(\nu) = s$. Decompose ∂D_3 by $\partial D_3 = s\mu_3 r\alpha_3 k\beta_3 s$, $\beta_3 \cap \partial M = k$. Let $\mu = u\alpha_1 v\nu s\beta_3^{-1} k$. We claim that μ is a separating path. Clearly α_1, ν and β_3^{-1} are simple paths by the property $CN(1)$ and $\alpha_1 \cap \nu = \{v\}$ and $\nu \cap \beta_3^{-1} = \{s\}$ by the property $CN(2)$. Also $\mu \cap \partial M = \{u, k\}$. It only remains to show that $u \neq k$. If $u = k$ then μ is a simple closed curve which meets ∂M in u only. Let $\mathcal{R}$ be the simply connected submap of M which has μ for its boundary. Then the only neighbours of $\mathcal{R}$ are D_1, D and D_3, contradicting

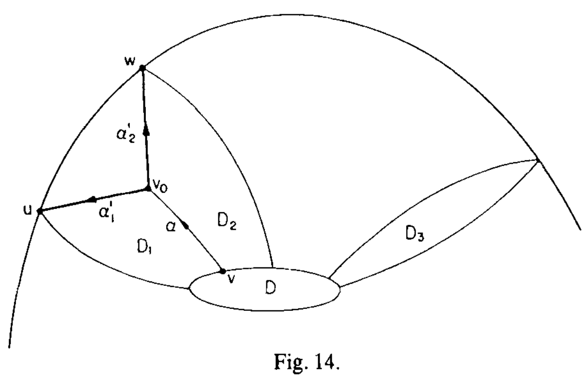


Fig. 14.

the property $CN(3)$. This completes the proof that μ is a separating path of the third kind. (See Fig. 13.)

2° $u \neq w$ (see Fig. 14). Then by the property $CN(2)$ we have that $\alpha_1 \cap \alpha_2^{-1}$ is connected. Let $\alpha = \alpha_1 \cap \alpha_2^{-1}$. Then we have decompositions $\alpha_1 = v\alpha v_0 \alpha_1' u$ and $\alpha_2^{-1} = v\alpha v_0 \alpha_2' w$. Let $\mu = u\alpha_1^{-1} v_0 \alpha_2 w$. Then μ is a separating path of the first kind, as is easy to show by the above arguments. This completes the proof of the Theorem.

§5. Proof of Theorem A

5.1. Theorem. *Let M be a connected, simply connected map which satisfies the Condition* W(4) *and contains more than one region. For every boundary region D of M denote by $i(D)$ the number of inner edges of D. Then M has a set $\mathcal{D}$ of boundary regions with $i(D) \leqq 3$ such that the following conditions hold*:

(I) *$\partial D \cap \partial M$ is connected for every $D \in \mathcal{D}$*;

(II) *if $i(D) = 3$ then ∂D has an inner vertex with valency 3 for every $D \in \mathcal{D}$*;

(III) $\Sigma_{D \in \mathcal{D}} [4 - i(D)] \geqq 6$.

Note that if M satisfies the condition G(3) then the theorem holds. We shall prove Theorem 5.1 by induction on $|M|$, the number of regions in M. Thus we assume

$\mathcal{H}_1$: M satisfies Condition W(4) but does not satisfy G(3).

$\mathcal{H}_2$: Every connected and simply connected proper submap of M with more than one region satisfies (I), (II) and (III) above.

We carry out the induction step by removing one of the boundary regions of M and then applying the induction hypothesis for the resulting map M'. The next lemma guarantees that M always has a boundary region D which is appropriate for this purpose.

5.2. Lemma. *Let M satisfy $\mathcal{H}_1$ and $\mathcal{H}_2$ and let μ be a separating path of M. Then $\mathrm{Sh}_M(\mu)$ contains a boundary region D of M with $i_M(D) \leqq 3$ such that* (I) *and* (II) *of* 5.1 *are satisfied.*

Proof. We may assume without loss of generality that μ is minimal in the sense that for no separating path τ does $\mathrm{Sh}_M(\tau) \subseteq \mathrm{Sh}_M(\mu)$ hold. Let $H = \mathrm{Sh}_M(\mu)$ and let $\mathcal{D}'$ be the set of boundary regions of H which results from $\mathcal{H}_2$. We shall show that $\mathcal{D}'$ has at most 2 elements which are not on the common boundary of H and M.

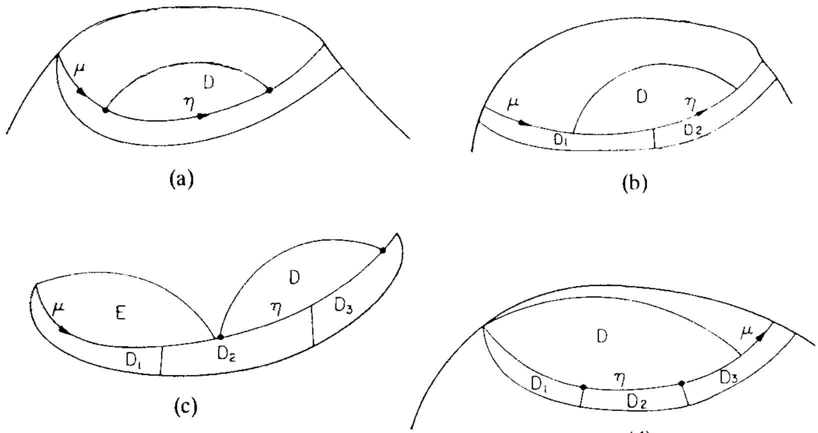


Fig. 15.

Let $D \in \mathscr{D}'$ and let $\eta = \partial D \cap \partial H$. We claim that we cannot have $\eta \subseteq \mu$. For assume $\eta \subseteq \mu$ (see Fig. 15).

By the definition of a minimal separating path, η satisfies exactly one of the following conditions:

(i) η consists of one edge (see Fig. 15(a)),

(ii) η consists of two edges with a vertex of valency 3 between them (see Figs. 15(b)) and 15(c)).

(iii) η consists of three edges with two vertices of valency 3 between them (see Fig. 15(d)).

In case (i) $d_M(D) \leqq 4$ and $d_M(D) = 4$ only if $i_M(D) = 3$. However in this case ∂D contains at least one vertex with valency 3, violating W(4). In case (ii), $d_M(D) \leqq 5$ and $d_M(D) = 5$ if $i_M(D) = 3$, and $d_M(D) = 4$ if $i_M(D) = 2$. However in the case $d_M(D) = 5$ ∂D contains at least 2 vertices with valency 3 while in the case $d_M(D) = 4$, ∂D contains at least one vertex with valency 3 (namely in μ). In both cases W(4) is violated. Case (iii) contradicts the minimality of μ. Consequently, $\eta \not\subseteq \mu$ and if $\eta \not\subseteq \partial M$ then either $o(\mu) \in \eta$ or $t(\mu) \in \eta$. Thus $\mathscr{D}'$ has at most 2 elements which are not on ∂M. If $|\mathscr{D}'| > 2$ then the number of elements of $\mathscr{D}$ on ∂M is at least $|\mathscr{D}'| - 2$, i.e., at least 1. On the other hand, if $|\mathscr{D}'| = 2$ then $i_H(D) = 1$ for every $D \in \mathscr{D}'$, hence $i_M(D) \in \{2,3\}$; see Fig. 16(a).

If $i_M(D) = 2$ for at least one region in $\mathscr{D}'$ then we are done. So assume $i_M(D_j) = 3$, $j = 1,2$, where $\mathscr{D}' = \{E_1, E_2\}$ (see Fig. 16(b)). Since ∂E_j has a vertex with valency 3 on its boundary (namely on μ) we are done in this case too.

5.3. Before coming to the general case we eliminate three special cases in the following three lemmas.

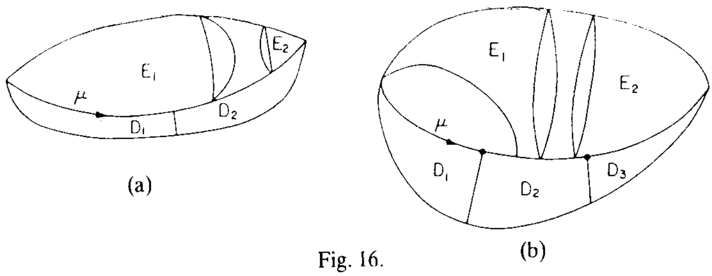


Fig. 16.

LEMMA. *Let D be a boundary region of M. If* $i(D)=1$ *then* (I), (II) *and* (III) *hold for M.*

PROOF. Let $M' = M\backslash\{D\}$. Then M' is connected and simply connected. We may assume that $|M'|>1$. By the induction hypothesis for M', a set $\mathscr{D}'$ of boundary regions of M' exists, which satisfies the conditions (I), (II) and (III). Let E be the unique neighbour of D. (See Fig. 17.)

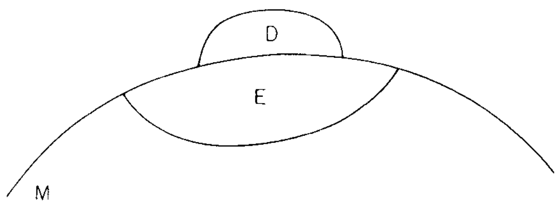


Fig. 17.

If $E \notin \mathscr{D}'$ then we may choose $\mathscr{D} = \mathscr{D}'$. If $E \in \mathscr{D}'$ then the contribution of E to the sum in (III) is at most 3. Since the same is true for D, the set $\mathscr{D} = \mathscr{D}'\backslash\{E\} \cup \{D\}$ satisfies the requirement of the theorem and the lemma holds.

5.4. LEMMA. *Let D be a boundary region of M. If* $i(D)=2$ *and the inner vertex v on* ∂D *has valency* 3, *then* (I), (II) *and* (III) *hold for M.*

PROOF. Let $M' = M\backslash\{D\}$ and let E_1 and E_2 be the neighbours of D in M (see Fig. 18). Let $\mathscr{D}'$ be the set of boundary regions of M' guaranteed by the

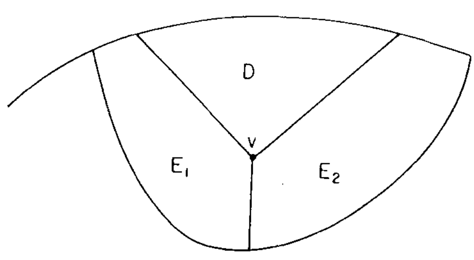


Fig. 18.

induction hypothesis and let $\mathscr{E} = \{E_j, j = 1,2 \mid i_{M'}(E_j) = 3\}$. Let $\mathscr{D} = \mathscr{D}' \cup \{D\} \setminus \mathscr{E}$. We claim that $\mathscr{D}$ satisfies the requirement of the Lemma. Condition (I) is satisfied by the construction of D and the induction hypothesis. Also (II) is satisfied by the induction hypothesis and the fact that v has valency 3. Hence if one of E_1 or E_2 belongs to $\mathscr{D}$ then its boundary contains a vertex with valency 3. Finally, to show (III) we note that $i_M(E) = i_{M'}(E)$ for $E \in \mathscr{D}' \setminus \{E_1, E_2\}$ and $i_M(E_j) = i_{M'}(E_j) + 1$ for $j = 1,2$. Consequently

$$\begin{aligned}\sum_{E \in \mathscr{D}} (4 - i_M(E)) &= \sum_{E \in \mathscr{D}' \setminus \mathscr{E}} (4 - i_M(E)) + 4 - i_M(D) \\ &= \sum_{E \in \mathscr{D}'} (4 - i_{M'}(E)) + 2 \\ &\geqq 6 - (i_M(E_1) - i_{M'}(E_1)) - (i_M(E_2) - i_{M'}(E_2)) + 2 \\ &\geqq 6 - 2 + 2 = 6\end{aligned}$$

as stated.

Finally, we mention one more case which can be dealt with immediately due to Lemma 5.2.

5.5. Lemma. *Let D be a boundary region of M. If $i_M(D) = 2$ and D has a neighbour E_1 with $i_M(E_1) = 2$, then* (I), (II) *and* (III) *hold for M.* (See Fig. 19.)

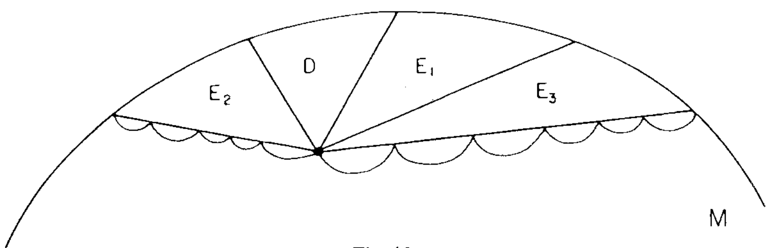


Fig. 19.

Proof. Let $M' = M \setminus \{D, E_1\}$. Let $\mathscr{D}'$ be the set of boundary regions of M' guaranteed by the induction hypothesis for M' and let E_2 and E_3 be the neighbours of D and E_1 which are different from D and E_1. If $i_{M'}(E_j) = 1$ for $j = 2$ or 3, we are done because then we may take $\mathscr{D} = \{D, E_1, E_j\}$. So assume $i_{M'}(E_j) \geqq 2$ for $j = 2,3$ and let $\mathscr{D} = (\mathscr{D}' \setminus \{E_2, E_3\} \cup \{D_1, E_1\})$. (I) and (II) for $\mathscr{D}$ follow from the corresponding properties for $\mathscr{D}'$.

Now

$$\sum_{E \in \mathscr{D}' \setminus \mathscr{E}} (4 - i_M(E)) = \sum_{E \in \mathscr{D}' \setminus \mathscr{E}} (4 - i_{M'}(E)) \geqq 6 - \Big(\sum_{E \in \mathscr{E}} 4 - i_{M'}(E) \Big).$$

Since $i_{M'}(E) \geqq 2$ for $E \in \mathscr{E}$, $4 - i_{M'}(E) \leqq 2$, hence $\Sigma_{E \in \mathscr{D} \setminus \mathscr{E}}(4 - i_M(E)) \geqq 6 - 4 = 2$. On the other hand, $i_M(D) = i_M(E_1) = 2$, hence $(4 - i_M(D)) + (4 - i_M(E_1)) = 4$. Consequently $\Sigma_{E \in \mathscr{D}}(4 - i_M(D)) \geqq 2 + 4 = 6$, as required.

Due to these lemmas, we shall make the following hypotheses:

$\mathscr{H}_3$: M has no boundary regions D with $i_M(D) = 1$.

$\mathscr{H}_4$: M has no boundary regions D with $i_M(D) = 2$ such that the inner vertex of ∂D has valency 3.

$\mathscr{H}_5$: M has no boundary regions D and E with $i_M(D) = i_M(E) = 2$ such that $\partial E \cap \partial D$ contains an edge.

In order to deal with the cases not covered by $\mathscr{H}_3$, $\mathscr{H}_4$ and $\mathscr{H}_5$, we have to introduce a special class of "one layer" maps around the region assumed by Lemma 5.2.

5.6. Definitions. Let M be a map and let D be a boundary region of M.

(a) For $k = 1,2,3$ call D a *k-corner of M* if

(i) $\partial D \cap \partial M$ is connected;

(ii) $i(D) = k$;

(iii) if $i(D) = 3$ then ∂D has an inner vertex with valency 3 in M.

(b) D is an 0-corner if M consists of D only. Denote the set of all the k-corners by $\mathrm{Cor}_k(M)$ and let

$$\mathrm{Cor}(M) = \mathrm{Cor}_1(M) \cup \mathrm{Cor}_2(M) \cup \mathrm{Cor}_3(M).$$

(c) Let M be a connected and simply connected map. Let S be a connected submap of M consisting of regions $D_1, \ldots, D_r$ and let M' be the map obtained from M by deleting all the regions of S.

S is called a *boundary strip* in M if the following hold:

(i) M' is connected (hence nonempty);

(ii) S is either simply connected or annular (see Figs. 20(a), (b) and (c));

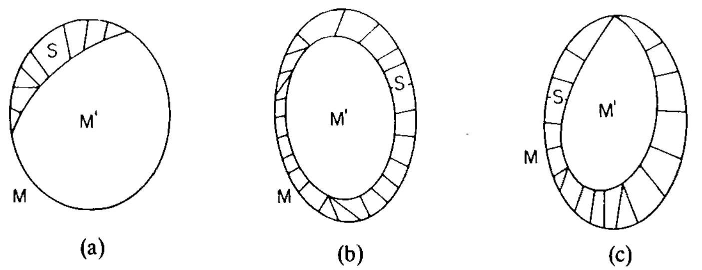


Fig. 20.

(iii) $\partial D_j \cap \partial M$ is connected and contains an edge $j = 1,\ldots,r$;
(iv) $\partial D_i \cap \partial M' \neq \emptyset$, $j = 1,\ldots,r$;
(v) $\partial D_j \cap \partial D_{j+1}$ contains an edge, $j = 1,\ldots,r-1$ (see Fig. 21).

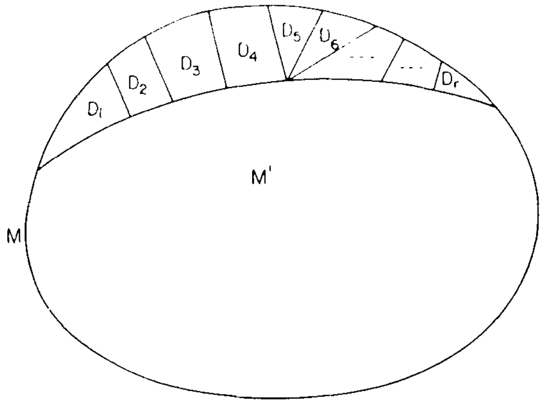


Fig. 21.

5.7. Lemma. *Let M be a connected, simply connected map satisfying $\mathcal{H}_i$, $i = 1,2,3,4,5$. Let Σ be a maximal boundary strip in M and let $M' = M \backslash \Sigma$. Let $E \in \mathrm{Cor}(M')$. Then either*

(a) $\partial E \cap \partial M' \subseteq \partial M$, *or*
(b) $\partial E \cap \partial M' \subseteq \partial \Sigma$ (see Fig. 22).

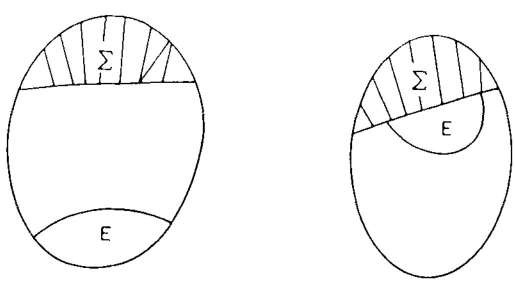
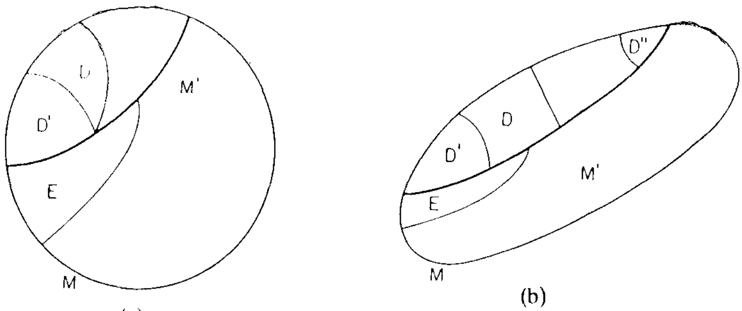


Fig. 22.

Proof. If Σ is annular, $\partial M' \subseteq \partial \Sigma$. Thus let us assume that Σ is simply connected. Let D' and D'' be the extreme regions of Σ. If both (a) and (b) fail to hold, then E has a common boundary edge with at least one of the regions D', D'', say with D', since $\partial E \cap \partial M'$ is connected. We claim that $\partial E \cap \partial \Sigma \subseteq \partial D'$. Assume not. Let D be the neighbour of D' in Σ (see Fig. 23).



(a) (b)

Fig. 23.

If $\partial D \cap \partial M' = v$, a single vertex, then the pair (D', D) violates $\mathcal{H}_5$ (see Fig. 23(a)). Hence $\partial D \cap \partial M'$ contains an edge. But then $\partial E \cap \partial D$ also contains an edge, a contradiction to $\mathcal{H}_4$. (See Fig. 23(b).) Thus $\partial E \cap \partial \Sigma \subseteq \partial D'$.

If M' contains a single region E (which means $E \in \mathrm{Cor}_0(M')$) then $\partial E \cap \partial \Sigma \subseteq \partial D'$ implies that D' is the single region of Σ, contradicting $\mathcal{H}_3$. (See Fig. 24.) For $E \in \mathrm{Cor}_k(M')$, $k = 1,2,3$, the map M'' obtained from M' by deleting E is connected. Therefore, adjunction of E to Σ yields a strip Σ_1 which properly contains Σ, in contradiction to the maximality of Σ. Consequently either (a) or (b) holds.

The following proposition is central to the main result.

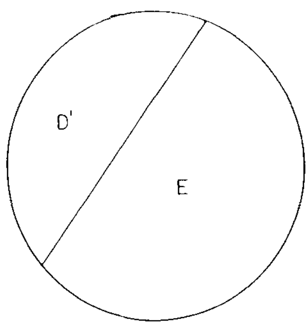


Fig. 24.

PROPOSITION. *Let M, Σ and M' be as in the previous lemma. Assume that if $\partial M \cap \partial M'$ is a single vertex v then it is not an inner vertex of $\partial E \cap \partial \Sigma$.* (This case will be considered in the next lemma.) (See Fig. 25.) *Let $E \in \mathrm{Cor}_k(M')$, $k > 0$ and assume that $\partial E \cap \partial M' \subseteq \partial \Sigma$. Then there exist distinct regions $F_1, \dots, F_l$ in M such that the following three conditions are satisfied:*

(a) $F_1 \in \mathrm{Cor}_{k_1}(M), \ldots, F_l \in \mathrm{Cor}_{k_l}(M)$;

(b) $\partial F_i \cap \partial M' \subseteq \partial E \setminus \{v_1, v_2\}$, *where* v_1 *and* v_2 *are the endpoints of* $\partial E \cap \partial \Sigma$ (see Fig. 26);

(c) $\Sigma_{i=1}^{l}(4 - i_M(F_i)) = \Sigma_{i=1}^{l}(4 - k_i) \geqq 4 - k$.

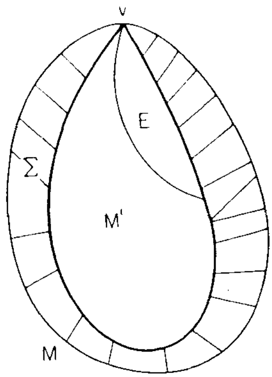


Fig. 25.

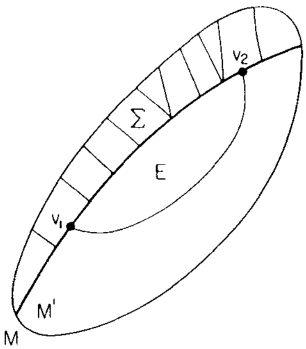


Fig. 26.

PROOF. Let Σ_1 be the minimal substrip of Σ which contains $\partial E \cap \partial M'$. Denote $C_{M'}(E) = 4 - i_{M'}(E)$ and $C_M(E) = \Sigma_{D \in \Sigma_1}[4 - i_M(D)]$. We propose to show that $C_M(E) \geqq C_{M'}(E)$.

Let $\mu = \partial E \cap \partial M'$. If v is a vertex on μ with valency $\geqq 4$ and v is neither $o(\mu)$ nor $t(\mu)$, then Σ contains a region $F \in \mathrm{Cor}_2(M)$ which contains v on its boundary. On the other hand if a_1 of the vertices of μ, which are neither $o(\mu)$ nor $t(\mu)$, have valency 3, then at least $a_1 - 1$ regions of Σ which contain these vertices on their boundary belong to $\mathrm{Cor}_3(M)$. (See Fig. 27.) Consequently if the number of non-extremal vertices of μ with valency 4 is b_1 then

$$C_M(E) \geqq a_1 - 1 + 2b_1. \tag{1}$$

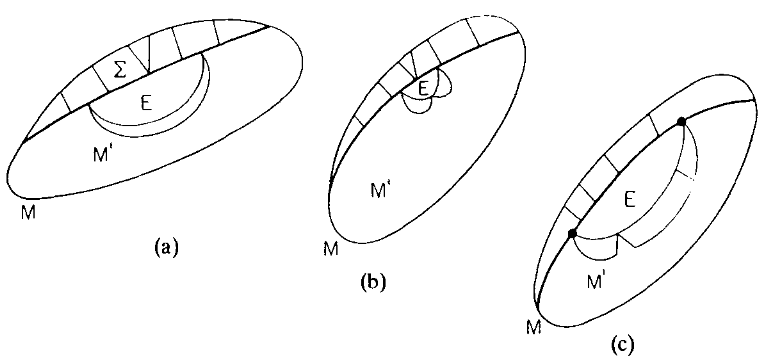


Fig. 27.

Denote by a_2 and b_2 the number of the remaining vertices on ∂E which have valency 3 and valency $\geqq 4$ respectively. Then because of the property $CN(1)$ we have

$$a_2 + b_2 = k + 1. \tag{2}$$

Assume now that $d(E) > 4$ and $d(E) \neq 8$. Then the W(4) condition implies

$$a_1 + a_2 + 2(b_1 + b_2) \geqq 9. \tag{3}$$

Combining (2) and (3) gives

$$a_1 + 2b_1 - 1 \geqq 7 - k - b_2. \tag{4}$$

Now if $k = 3$ then $b_2 \leqq 3$; if $k = 2$ then $b_2 \leqq 3$ and if $k = 1$ then $b_2 \leqq 2$. Consequently in all cases $b_2 \leqq 3$, hence

$$7 - k - b_2 \geqq 4 - k = C_{M'}(E). \tag{5}$$

Substitution of (4) and (5) in (1) gives $C_M(E) \geqq C_{M'}(E)$ as required.

Assume now $d(E) = 4$. Then $k = 1$, or 2. Since all the vertices on ∂E have valency 4 we get $d_1 = 0$, $a_2 = 0$ and $b_1 \geqq 1$ in case $k = 2$ and $b_1 \geqq 2$ in case $k = 1$. Accordingly $C_M(E) \geqq C_{M'}(E)$ holds in all cases, as required. (See Fig. 27(b).) Finally, assume $d(E) = 8$. (See Fig. 27(c).) Then $a_1 + b_1 = 8 - (k+1) = 7 - k$. Consequently, by (1),

$$\begin{aligned} C_M(E) \geqq a_1 - 1 + 2b_1 &= a_1 + b_1 + (b_1 - 1) \\ &= 7 - k + (b_1 - 1) \\ &= 6 - k + b_1 > 4 - k \\ &= C_{M'}(E), \end{aligned}$$

i.e. $C_M(E) > C_{M'}(E)$, as required.

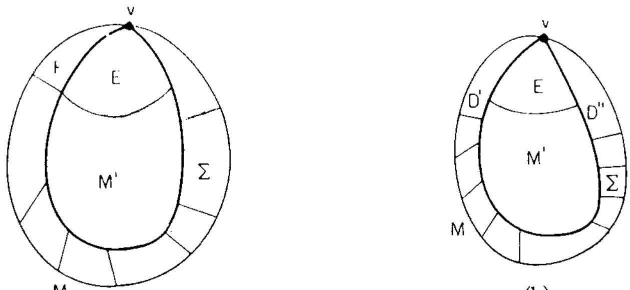
5.8. Lemma. *Let M, Σ and M' be as in the previous lemma. Assume that $\partial M' \cap \partial M$ consists of a unique vertex v. Let $E \in \mathrm{Cor}_k(M')$ and assume that $v \in \partial E$ and v is not an endpoint of $\partial E \cap \partial M'$* (see Fig. 28). *Then $k \geqq 2$ and one of the following holds*:

(a) *M has a region $F \in \mathrm{Cor}_2(M)$ with $v \in \partial F$ such that $\partial F \cap \partial M'$ contains an edge and is contained in ∂E* (see Fig. 28(a)), *or*

(b) $\mathrm{Cor}(M')$ *has a subset $\mathcal{D}'$ not containing E, such that*

$$\sum_{D \in \mathcal{D}'} (4 - i(D)) \geqq 6 \tag{*}$$

is satisfied. (See Fig. 28(b).)



(a) (b)

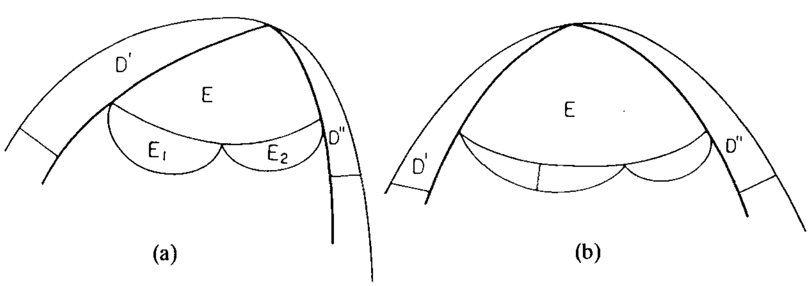
Fig. 28.

PROOF. Let D' and D'' be the extreme regions of Σ. Then by $\mathscr{H}_4$ and $\mathscr{H}_5$ we have $\partial E \cap \partial M' \subseteq \partial D' \cup \partial D''$ (see Fig. 29). Consequently, if $E_1,\dots,E_k$ are the neighbours of E in M' then

(**) the neighbours of E in M are $E_1,\dots,E_k, D'$ and D''.

We claim that $k \geqq 2$. First, $k > 0$, i.e. E cannot be the single region of M' for the following reason: Because of the $CN(2)$ property, Σ contains at least three regions, contradicting (**). Hence $k \geqq 1$. But if $k = 1$, then $d_M(E) = 3$ by (**), contradicting W(4). Thus $k \geqq 2$, i.e. $k = 2$ or $k = 3$. Assume (a) does not hold. We show that (b) necessarily holds. Let M'' be the map obtained from M' by deleting E. Then M' contains at least 2 regions, therefore $\mathrm{Cor}(M'')$ has a subset $\mathscr{D}'$ such that (*) holds by $\mathscr{H}_2$. We propose to show that $\mathscr{D}' \subseteq \mathrm{Cor}(M')$. To this end it is enough to show that $E_1,\dots,E_k$ do not belong to $\mathrm{Cor}(M'')$. Assume that one of the E_j belongs to $\mathrm{Cor}_l(M'')$, $l \geqq 1$. We distinguish two cases:

Case 1. $k = 2$. Then by (**) $d_M(E) = 4$, hence all the boundary vertices of E have valency at least 4. Consequently $\partial E \cap \partial M'$ consists of a single vertex, since (a) does not hold (see Fig. 29(a)).

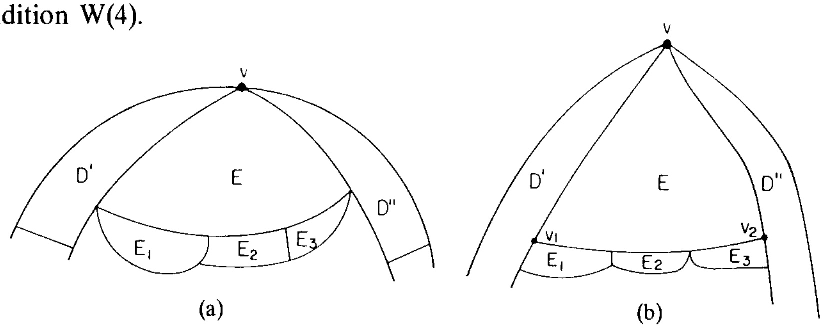


(a) (b)

Fig. 29.

Hence the neighbours of E_j in M are those which are in M'' together with E. This implies $d_M(E_j) = l+1$. Since $d_M(E_j) \geqq 4$, we have $l = 3$, i.e. $E_j \in \mathrm{Cor}_3(M'')$ and all the vertices on ∂E_j have valency at least 4. But by the definition of $\mathrm{Cor}_3(M'')$, ∂E_j contains a vertex with valency 3, a contradiction. Consequently, $E_j \notin \mathrm{Cor}(M'')$.

Case 2. $k = 3$. (See Fig. 29(b).) Then by (**), $d_M(E) = 5$. We claim that $\partial E_j \cap \partial M'$ consists of at most one vertex for $j = 1,2,3$. (See Fig. 30(a).) This is clear for $j = 2$. On the other hand if $\partial E_j \cap \partial M'$ contains an edge for $j = 1$ or $j = 3$, then since (a) does not hold, this edge necessarily has an endpoint v_j with valency 3 (see Fig. 30(b)). But then since $v_j \in \partial E$, by the definition of $\mathrm{Cor}_3(M'')$, ∂E contains at least 2 vertices with valency 3. This however contradicts the condition W(4) as $d_M(E) = 5$. Thus $\partial E_j \cap \partial M'$ consists of at most one vertex. If $j = 2$ and $E_j \in \mathrm{Cor}_l(M'')$ then the neighbours of E_j in M are those in M'' together with E_1 (or E_3) and E, hence $d_M(E_j) = l+2$. But then $l = 2$ or 3 and ∂E_j contains at least $l-2+1$ vertices with valency 3. This however violates the condition W(4).



(a) (b)

Fig. 30.

If $j = 1$ (or $j = 3$) then the neighbours of E_j are those in M'' together with E, as $\partial E_j \cap \partial M'$ does not contain an edge. Accordingly $d_M(E_j) = l+1$. Thus $l = 3$, as $d_M(E_j) \geqq 4$ and $l \geqq 3$. But then $d_M(E_j) = 4$ and ∂E_j contains a vertex with valency 3, by the definition of $\mathrm{Cor}_3(M'')$. This again violates the condition W(4). Consequently $E_j \notin \mathrm{Cor}(M'')$ and the lemma is proved.

5.9. We now turn to the proof of the theorem. By Lemmas 5.3–5.5, we may assume that $\mathcal{H}_1, \ldots, \mathcal{H}_5$ hold. By Lemma 5.2 M has a maximal boundary strip Σ. Let M' be the map obtained from M by deleting Σ. Assume first that M' contains more than one region. Then, again by the induction hypothesis $\mathrm{Cor}(M')$ has a subset $\mathcal{D}'$ such that

$$\sum_{D \in \mathcal{D}'} [4 - i_{M'}(D)] \geqq 6. \tag{1}$$

By Lemma 5.7 we can write $\mathscr{D}' = \mathscr{D}_1' \dot{\cup} \mathscr{D}_2'$ where $\partial E \cap \partial M \subseteq \partial M$ for every $E \in \mathscr{D}_1'$ and $\partial E \cap \partial M' \subseteq \partial \Sigma$ for every $E \in \mathscr{D}_2'$. By Lemma 5.8 we may assume that if $\partial M' \cap \partial M$ consists of a unique vertex then no element E of $\mathscr{D}_2'$ contains v on its boundary as an endpoint of $\partial M' \cap \partial E$. Consequently, by Proposition 5.7, for every $E \in \mathscr{D}_2' \cap \mathrm{Cor}_k(M')$ there are regions $F_j \in \mathrm{Cor}_{k_j}(M)$, $j = 1, \ldots, l$ such that if $\mu = \partial E \cap \partial \Sigma$ then

$$\partial F_j \cap \partial \Sigma \subseteq \mu \backslash \{0(\mu), t(\mu)\}, \tag{2}$$

$$\sum_{j=1}^{l} (4 - i_M(F_j)) \geqq 4 - k. \tag{3}$$

Denote by $\mathscr{D}_2$ the set of all the F_j we get from $\mathscr{D}_2'$ this way. Let $\mathscr{D} = \mathscr{D}_1' \cup \mathscr{D}_2$. Then $\mathscr{D} \subseteq \mathrm{Cor}(M)$ and as $\mathscr{D}_1' \cap \mathscr{D}_2 = \varnothing$ we get

$$\begin{aligned}\sum_{D \in \mathscr{D}} (4 - i_M(D)) &= \sum_{D \in \mathscr{D}_1'} (4 - i_{M'}(D)) + \sum_{D \in \mathscr{D}_2} (4 - i_M(D)) \\ &\geqq \sum_{D \in \mathscr{D}_1} (4 - i_{M'}(D)) + \sum_{D \in \mathscr{D}_2} (4 - i_{M'}(D)) \qquad \text{by (2) and (3)} \\ &\geqq 6 \qquad \text{by (1)},\end{aligned}$$

i.e.

$$\sum_{D \in \mathscr{D}} (4 - i_M(D)) \geqq 6. \tag{$*$}$$

Finally, assume $|M'| = 1$. Then Σ contains at least three regions by $\mathscr{H}_3$, $\mathscr{H}_4$ and $\mathscr{H}_5$ (see Fig. 31(a)). If Σ contains exactly 3 regions then $E \in \mathrm{Cor}_3(M)$, by $CN(3)$,

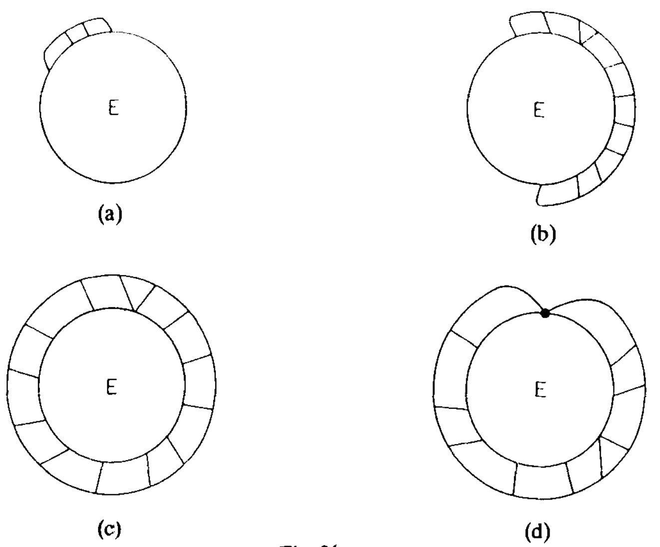


Fig. 31.

hence $\mathscr{D}$ = {the regions of M} satisfies (∗). On the other hand, if Σ contains more than 3 regions (see Figs. 31(b) and (c)) then we distinguish three cases:

Case 1. Σ is simply connected. Let D' and D'' be the extreme regions of Σ. Then $d_M(D') = d_M(D'') = 2$ and if Σ contains no more regions F with $d_M(F) = 2$ then $d_M(F) = 3$ for every $F \in \Sigma \backslash \{D', D''\}$.

Case 2. Σ is an annular map and $\partial E \cap \partial M = \emptyset$. Then the condition W(4) for E immediately implies (∗).

Case 3. Σ is annular and $\partial E \cap \partial M$ consists of one vertex. Then the arguments of Case 1 hold.

This completes the proof of the Theorem.

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